7/05/2014

Lec 23

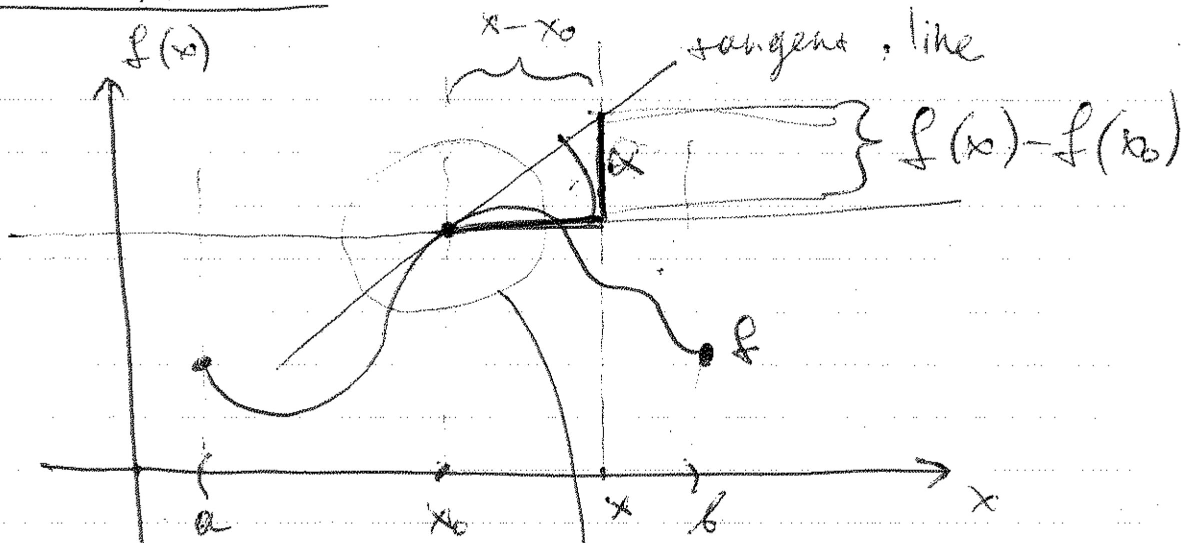
Differentiation.

Suppose $x_0 \in (a,b)$, $f:(a,b) \to \mathbb{R}$

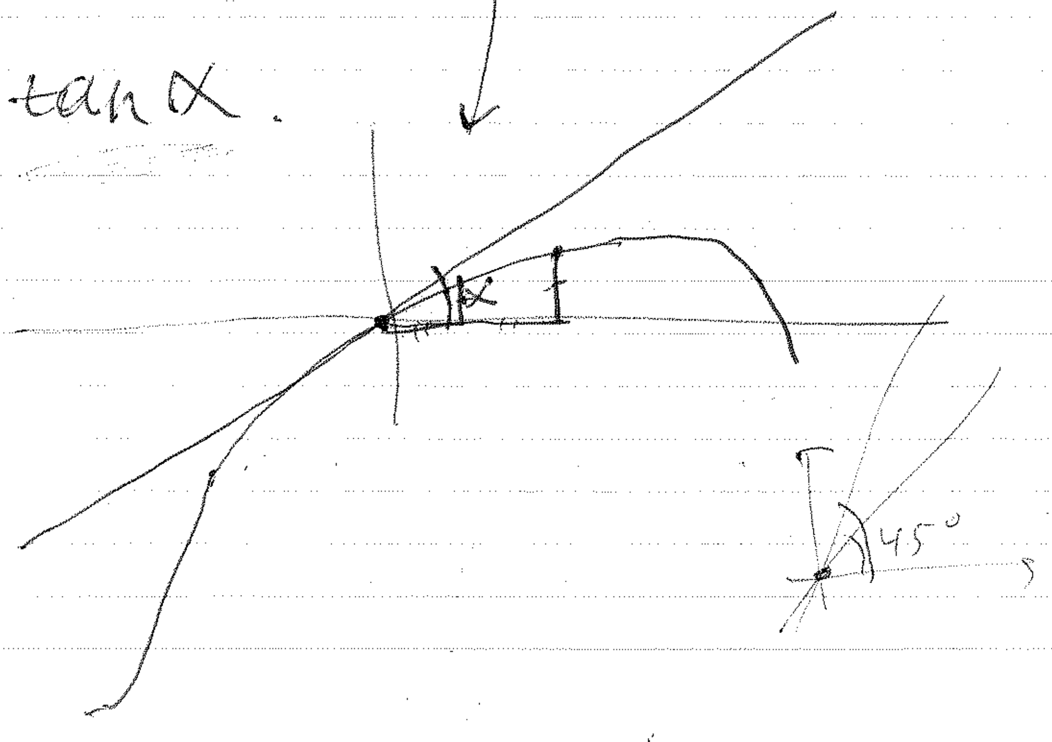
Def. The <u>derivative</u> of f at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If these limits exist, then f is <u>differentiable</u> at x_0.



$f'(x_0) = \tan\alpha$.



Equation of the tangent line:

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

Proposition. If f is differentiable at x_0, then f is continuous at x_0.

Proof

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}(x-x_0) + f(x_0)\right) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\lim_{x\to x_0}(x-x_0)$$

$$+\lim_{x\to x_0} f(x_0) = f'(x_0)\cdot 0 + f(x_0) = f(x_0)$$

(because all three limits exist) $\square$

Example $f(x) = |x|$ is uniformly contin. on $[-1,1]$ (exercise), but not diff. at 0 (exercise).

Example Let's compute $f'(x)$ for $f(x) = x^3$. By def.,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{\cancel{x^3} + h^3 + 3h^2x + 3hx^2 \ldots}{h}$$

$$= \lim_{h \to 0} \left(\underset{\searrow 0}{h^2} + \underset{\searrow 0}{3hx} + 3x^2 \right) = 3x^2 .$$

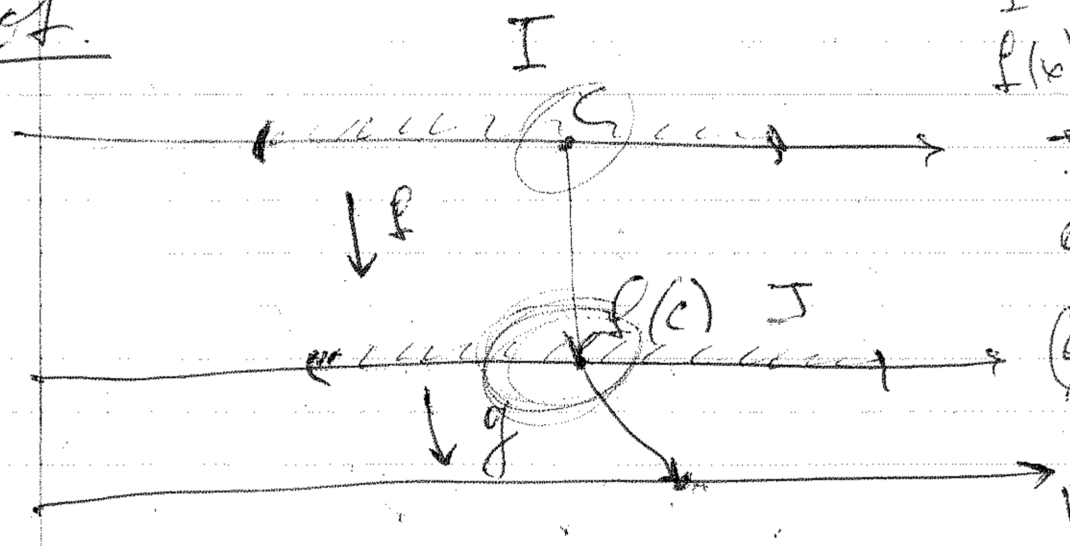
<u>Theorem</u> (chain rule). Suppose I, J a[re] open intervals in $\mathbb{R}$; take $f: I \to J$ an[d] $g: J \to \mathbb{R}$. Suppose f is diff. at $c \in I$ and g is diff. at $f(c) \in$ [J]. Then the function $g \circ f : I \to \mathbb{R}$ is diff. at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

(Note: $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x)) .)$$

<u>Proof.</u>



$I = (0, 2)$

$f(x) = x^2$

$J = (0, 4)$

$g(y) = y^3$

$(g \circ f)(x) = x^6$

Denote $d = f(c)$.
Define $\alpha: I \to \mathbb{R}$ and $\beta: J \to \mathbb{R}$ by

$$\alpha(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} - f'(c), & x \neq c, \\ 0, & x = c, \end{cases}$$

$$\beta(y) = \begin{cases} \frac{g(y)-g(d)}{y-d} - g'(d) & y \neq d, \\ 0, & y = d. \end{cases}$$

Note that α and β are continuous.
Indeed, α is obviously contin away from c. Also, $\lim_{x\to c} \alpha(x) = \alpha(c) = 0$, by def. β - similarly.
Setting $y = f(x)$, we find for $x \neq c$

$$(g\circ f)(x) - (g\circ f)(c) = g(f(x)) - g(f(c))$$
$$= g(y) - g(d) = (y-d)(g'(d) + \beta(y))$$
$$= (f(x) - f(c))(g'(d) + \beta(y))$$
$$= (x-c)(f'(c) + \alpha(x))(g'(d) + \beta(y)).$$

Therefore,

$$\frac{(g\circ f)(x)-(g\circ f)(c)}{x-c} = (f'(c)+\alpha(x)) \times (g'(d)+\beta(y))$$

Now take lim:

$$(g\circ f)'(c) = \lim_{x\to c}\frac{(g\circ f)(x)-(g\circ f)(c)}{x-c}$$

$$= \cancel{\lim}\,\left(f'(c)+\lim_{x\to c}\alpha(x)\right) \times \left(g'(f(c))+\lim_{x\to c}\beta(f(x))\right)$$

$$= \left(f'(c)+\overbrace{\alpha(c)}^{0}\right)\left(g'(f(c))+\overbrace{\beta(f(c))}^{0}\right)$$

$$= f'(c)\cdot g'(f(c)).$$ $\square$

<u>Theorem</u>. Take $f, g: I \to \mathbb{R}$, diff. at $x \in I$. Then $f+g$, fg, $\frac{f}{g}$ are diff at x (last one if $g \neq 0$) and

1) $(f+g)'(x) = f'(x)+g'(x)$,

2) $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$.

3) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

Proof Exercise $\square$

1. ~~Mathematicians don't include authors?~~
2. ~~Make pictures pretty~~